Derivation of the spatio-temporal model equations for the thermoacoustic resonator.

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Abstract

We derive the model equations describing the thermoacoustic resonator, that is, an acoustical resonator containing a viscous medium inside. Previous studies on this system have addressed this sytem in the frame of the plane-wave approximation, we extend the previous model to by considering spatial effects in a large aperture resonator. This model exhibits pattern formation and localized structures scenario.

1 Derivation of the model

The physical system we consider is an acoustic resonator composed of two parallel walls containing a viscous fluid medium inside (e.g. glycerine). The case of sound beam propagating in viscous media is characterized by a strong absorption, being the sound velocity dependent on the fluid temperature, which results in an addicional nonlinearity mechanism from thermal origin.

1.1 Sound propagation in a viscous medium

The propagation of sound in a viscous, heat-conducting medium is described by the equations of mechanics of continuum media [1], *i.e.* the continuity equation

$$\frac{\partial \rho}{\partial t} + \rho_0 \boldsymbol{\nabla} \cdot \mathbf{v} = 0, \tag{1}$$

the momentum transfer equation

$$\frac{\partial \mathbf{v}}{\partial t} + \frac{1}{\rho_0} \boldsymbol{\nabla} p = \rho_0 \mu \nabla^2 \mathbf{v} + \left(\mu_B + \frac{\mu}{3} \right) \boldsymbol{\nabla} \left(\boldsymbol{\nabla} \cdot \mathbf{v} \right), \tag{2}$$

and the heat transport equation

$$\rho_0 c_p \frac{\partial T}{\partial t} - \kappa \nabla^2 T = \mu \frac{\partial v_i}{\partial x_k}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial v_l}{\partial x_l}\right) + \mu_B \left(\boldsymbol{\nabla}\cdot\mathbf{v}\right)^2, \qquad (3)$$

where ρ and p represent, respectively, the density and acoustic pressure deviations from the equilibrium state (denoted by ρ_0 and p_0), $\mathbf{v}$ is the fluid particle velocity. The constants μ, μ_B, κ and c_p are the coefficients of shear and bulk viscosity, the thermal conductivity and the specific heat at constant pressure of the fluid, respectively. In Eq.(3), T denotes the perturbation of the medium temperature due to the acoustic wave, and x_i the spatial coordinates.

Equations (1)-(3) must be complemented with the equation of state relating the medium density with the other variables. For a linear medium (small amplitudes) it obeys the simple form $p = c_0^2\rho$. In a highly absorbing medium, the thermal effects are dominant and the hydrodynamic nonlinearity can be neglected. Furthermore, the speed of sound depends on the average value (over one period) of the temperature $T_{av} = T_0 + \bar{T}$, where T_0 is the temperature at equilibrium, so the equation of state takes the form $\rho = p/c_0^2(\bar{T})$. Expanding at leading order it follows

$$\rho = \frac{1}{c_0^2}p - \frac{\gamma_p}{c_0^2}p\bar{T}. \qquad (4)$$

where $\gamma_p = (2/c_0)\left(\frac{\partial c}{\partial T}\right)_{T=T_{eq}}$

We assume that the fields can be described by quasi-plane waves, i.e. they take the form

$$p = \frac{1}{2}P(\mathbf{r},t)e^{i\omega t} + c.c., \qquad (5a)$$

$$\mathbf{v} = \frac{1}{2}\mathbf{u}(\mathbf{r},t)e^{i\omega t} + c.c., \qquad (5b)$$

$$T = \frac{1}{2}T(\mathbf{r},t)e^{i\omega t} + c.c. + \bar{T}(\mathbf{r},t), \qquad (5c)$$

where a slowly changing temperature $\bar{T}$, responsible for the thermal self-action of the waves on $v = v_z$, is added. Taking into account the slowly varying envelope condition

$$\frac{\partial X_i}{\partial x_j} << kX_i,\ \frac{\partial X_i}{\partial t} << \omega X_i, \qquad (6)$$

and the linear impedance relation relation $v = p/\rho_0 c_0$, the equation for $\bar{T}$ is obtained by averaging of Eq.(3)

$$\frac{\partial \bar{T}}{\partial t} - \chi\nabla^2\bar{T} = \frac{k^2 b}{2\rho_0^2 c_0^2}\left|P\right|^2, \qquad (7)$$

where $\bar{T}$ represents the average temperature perturbation caused by the wave energy dissipation. In Eq.(7) the new parameter b is defined as $b = (\mu_B + 4\mu/3)/\rho_0 c_p$, and $\chi = \kappa/\rho_0 c_p$ is the thermal diffusivity.

On the other side, substituting Eqs.(5) in Eqs.(1) and (2) it follows that, in the parabolic approximation, the pressure field is described by [1]

$$c_0 \frac{\partial P}{\partial z} + \frac{\partial P}{\partial t} + \frac{i c_0^2}{2\omega} \nabla_\perp^2 P + c_0 a P = i\omega \frac{\gamma_p}{2} P \bar{T} \tag{8}$$

where $a = b\omega^2/2\rho_0 c_0^3$ and $\nabla_\perp^2$ is a laplacian operator acting on transverse coordinates.

1.2 Introducing the resonator

1.2.1 Modal equations

In a resonator, the reflection at the boundaries imply the presence of counter-propagating traveling waves,and the acoustic field p can be expressed in terms of forward and backward components

$$P = P_F(\mathbf{r}, t) e^{i(\omega t - kz)} + P_B(\mathbf{r}, t) e^{i(\omega t + kz)}. \tag{9}$$

whose amplitudes are assumed to vary slowly in space and time (compared with the evolution described by the exponential factors).

Substitution of expansion (9) into Eq.(7), leads to the following temperature distribution

$$\bar{T} = T_h(\mathbf{r}, t) + T_g(\mathbf{r}, t) e^{i2kz} + T_g^*(\mathbf{r}, t) e^{-i2kz}, \tag{10}$$

where T_0 is the homogeneous component and T_1 is the amplitudes of the thermal grating induced by the acoustic wave. The acoustic resonator containing a viscous fluid medium inside (e.g. glycerine), has been previously addressed [2]in the frame of plane-wave approximation, we extend now the previous model by considering the sound diffraction and temerature diffusion.

Projecting on the different longitudinal modes, the following equations for the amplitudes are obtained:

$$\begin{aligned} \frac{\partial P_F}{\partial t} + c_0 \frac{\partial P_F}{\partial z} - i \frac{c_0^2}{2\omega} \nabla_\perp^2 P_F + c_0 a P_F &= i\omega \frac{\gamma_p}{2} \left(T_h P_F + T_g^* P_B \right), && \text{(11a)} \\ \frac{\partial P_B}{\partial t} - c_0 \frac{\partial P_B}{\partial z} - i \frac{c_0^2}{2\omega} \nabla_\perp^2 P_B + c_0 a P_B &= i\omega \frac{\gamma_p}{2} \left(T_h P_B + T_g P_F \right), && \text{(11b)} \end{aligned}$$

$$\begin{aligned} \frac{\partial T_h}{\partial t} - \chi \nabla_\perp^2 T_h &= \frac{k^2 b}{2\rho_0^2 c_0^2} \left(|P_F|^2 + |P_B|^2 \right), && \text{(12a)} \\ \frac{\partial T_g}{\partial t} + 4\chi k^2 T_g - \chi \nabla_\perp^2 T_g &= \frac{k^2 b}{2\rho_0^2 c_0^2} P_F^* \, P_B, && \text{(12b)} \end{aligned}$$

where $a = \omega^2 \left(\mu_B + 4\mu/3 \right) / (2\rho_0 c_0^3)$.

1.2.2 Boundary Conditions.

We consider an acoustic resonator with length L, filled with a viscous medium, and bounded by reflecting surfaces, with intensity reflection coefficients $\mathcal{R}$, located at $z = 0$ and $z = L$. We assume that one of the surfaces, vibrating at frequency ω, acts as ultrasonic source. In accordance with (9), boundary conditions satisfy

$$\begin{aligned} P_F(\mathbf{r}_\perp, z &= 0; t) = \sqrt{\mathcal{R}} P_B(\mathbf{r}_\perp, z = 0; t) + P_{in}, &(13) \\ P_B(\mathbf{r}_\perp, z &= L, t) = \sqrt{\mathcal{R}} P_F(\mathbf{r}_\perp, z = L; t) e^{-i\delta}, &(14) \end{aligned}$$

where the detuning parameter δ has been defined as

$$\delta = 2m\pi - 2kL = 2L\frac{\omega_c - \omega}{c_0}, \tag{15}$$

where m is an integer, selected in order to be ω_c the cavity frequency that lies nearest to the driving frequency ω.

1.2.3 Mean Field Limit.

When the reflectivity of the boundary walls is high, $\mathcal{R} \to 1$, the resulting model can be greatly symplified adopting the mean field limit. Under this assumption, the two counterpropagating fields are approximately constant and equal along the cavity (i.e. there is no spatial modulation of the field envelopes), and one can assume that

$$\bar{P}_F(\mathbf{r}_\perp, t) = \bar{P}_B(\mathbf{r}_\perp, t) \equiv P(\mathbf{r}_\perp, t) \tag{16}$$

where the overbar means the spatial average over the longitudinal coordinate z,

$$\bar{f} = \frac{1}{L}\int_0^L dz\ f(z). \tag{17}$$

We also consider that the frequency detuning between the driving and a cavity mode is small, $2L\left(\omega_c - \omega\right)/c << \pi$, which allows to write the boundary condition Eq.(14) as

$$P_B(\mathbf{r}_\perp, L, t) \approx \sqrt{\mathcal{R}} P_F(\mathbf{r}_\perp, L, t)(1 - i\delta). \tag{18}$$

Averaging Eqs.(11), with

$$\frac{1}{L}\int_0^L \frac{\partial P_{f,b}}{\partial z} dz = \frac{P_{f,b}\left(L\right) - P_{f,b}\left(0\right)}{L}.$$

leads to

$$\begin{aligned} \frac{\partial \bar{P}_F}{\partial t} + \frac{c_0}{L}\left[P_F\left(L\right) - P_F\left(0\right)\right] - i\frac{c_0^2}{2\omega}\nabla_\perp^2 \bar{P}_F + c_0 a \bar{P}_F &= i\omega\frac{\gamma_p}{2}\left(T_h P_F + T_g^* P_B\right), \\ \frac{\partial \bar{P}_B}{\partial t} - \frac{c_0}{L}\left[P_B\left(L\right) - P_B\left(0\right)\right] - i\frac{c_0^2}{2\omega}\nabla_\perp^2 \bar{P}_B + c_0 a \bar{P}_B &= i\omega\frac{\gamma_p}{2}\left(T_h P_B + T_g P_F\right), \end{aligned}$$

they can be reduced just to one equation

$$\partial_t P = -\frac{c_0}{2\sqrt{\mathcal{T}}L}(\mathcal{T}+i\delta)P_T+\frac{c_0}{2L}P_{in}+i\frac{c_0^2}{2\omega}\nabla_\perp^2 P-c_0 aP+i\omega\frac{\gamma_p}{2}\left(\bar{T}_h+\frac{\bar{T}_g+\bar{T}_g^*}{2}\right)P, \tag{19}$$

where the averaged temperature components as defined as $\bar{T}_i = \mathcal{L}T_i$ and we have used the property $\overline{(fg)} = \bar{f}\ \bar{g}$ [3], valid in the mean field case. The transmitted pressure can be assumed to be

$$p_T(t) = \sqrt{\mathcal{T}}P(\mathbf{r}_\perp;t) \tag{20}$$

and Eq.(19) is written as

$$\partial_t P = -\Gamma(1+i\theta)P + P_{in} + i\frac{c_0^2}{2\omega}\nabla_\perp^2 P + i\omega\frac{\gamma_p}{2}\left(\bar{T}_h+\frac{\bar{T}_g+\bar{T}_g^*}{2}\right)P, \tag{21}$$

where the parameters

$$\Gamma \equiv a\ c_0 + \mathcal{T}\frac{c_0}{2L}, \tag{22}$$

$$\theta \equiv \frac{\omega_c-\omega}{\Gamma}, \tag{23}$$

$$P_{in} \equiv \frac{c_0}{2L}p_{in}, \tag{24}$$

have been introduced. As well, under the same assumptions, Eqs.(12) read

$$\frac{\partial\bar{T}_h}{\partial t} - \chi\nabla_\perp^2\bar{T}_h = \frac{k^2 b}{\rho_0^2 c_0^2}\left|P\right|^2; \tag{25a}$$

$$\frac{\partial\bar{T}_g}{\partial t} + 4k^2\chi\bar{T}_g - \chi\nabla_\perp^2\bar{T}_g = \frac{k^2 b}{2\rho_0^2 c_0^2}\left|P\right|^2, \tag{25b}$$

1.3 The original model

The mean-field model for the thermoacoustic resonator can be written as

$$\partial_t\bar{P} = -\frac{c_0}{2L}(\mathcal{T}+i\delta)\bar{P}+\frac{c_0}{2L}\bar{P}_{in}+i\frac{c_0^2}{2\omega}\nabla_\perp^2\bar{P}-c_0 a\bar{P}+i\frac{\omega\gamma_p}{2}\left(\bar{T}_h+\bar{T}_g\right)\bar{P}, \tag{26a}$$

$$\partial_t\bar{T}_h = -\gamma_h\bar{T}_h+\chi\nabla_\perp^2\bar{T}_h+\frac{k^2 b}{\rho_0^2 c_0^2}\left|\bar{P}\right|^2, \tag{26b}$$

$$\partial_t\bar{T}_g = -4k^2\chi\bar{T}_g+\chi\nabla_\perp^2\bar{T}_g+\frac{k^2 b}{2\rho_0^2 c_0^2}\left|\bar{P}\right|^2, \tag{26c}$$

where $\bar{T}_g$ has already been assumed to be real. We note that the model is highly reminiscent of that for a coherently driven optical cavity filled with a non-instantaneous (and nonlocal) Kerr medium [4]. Unlike its optical analog, in this thermoacoustic model there are two nonlinearities (owed to $\bar{T}_h$ and to $\bar{T}_g$);

notice however that for $\gamma_h = 4k^2\chi$ both nonlinearities behave as a single one (a single temperature field $\bar{T}_h + \bar{T}_g$ can be defined so that its evolution and that of the pressure just depend on themselves in that case). Defining the following quantities

$$t_{\rm p}^{-1} = \frac{c_0\mathcal{T}}{2L} + c_0 a,\ \Delta = t_{\rm p}\left(\omega_c - \omega\right),\ \tau_{\rm p} = \gamma_{\rm h} t_{\rm p},\ \tau_{\rm g} = \frac{\gamma_{\rm h}}{4k^2\chi} \tag{27}$$

$$\tau = \gamma_{\rm h} t,\ \nabla^2 = \frac{t_{\rm p} c_0^2}{2\omega}\nabla_\perp^2,\ D = \frac{\chi}{\gamma_{\rm h} t_{\rm p}}\frac{2\omega}{c_0^2},\ P_{\rm in} = t_{\rm p}\frac{c_0}{2L}\sqrt{\frac{\gamma_p \omega \tau_{\rm p} k^2 b}{4\gamma_{\rm h}\rho_0^2 c_0^2}}\bar{P}_{in} \tag{28}$$

$$P = \sqrt{\frac{\gamma_p \omega \tau_{\rm p} k^2 b}{4\gamma_{\rm h}\rho_0^2 c_0^2}}\bar{P},\ H = \frac{\gamma_p \omega \tau_{\rm p}}{2}\bar{T}_{\rm h},\ G = \frac{\gamma_p \omega \tau_{\rm p}}{2}\bar{T}_{\rm g}, \tag{29}$$

the model equations, Eqs.(26) become

$$\tau_{\rm p}\partial_\tau P = -\left(1 + i\Delta\right)P + P_{\rm in} + i\nabla^2 P + i\left(H + G\right)P, \tag{30a}$$

$$\partial_\tau H = -H + D\nabla^2 H + 2\left|P\right|^2, \tag{30b}$$

$$\partial_\tau G = -\tau_{\rm g}^{-1} G + D\nabla^2 G + \left|P\right|^2. \tag{30c}$$

We consider a resonator with high quality plates ($\mathcal{T} = 0.1$), separated by $L = 5$ cm, driven at a frequency $f = 2$MHz and containing glycerine at 10^oC. Under these conditions the medium parameters are $c_0 = 2\times10^3$m s^{-1}, $\alpha_0 = 10$ m^{-1}, $\rho_0 = 1.2\times10^3$kg m^{-3}, $c_p = 4\times10^3$J kg^{-1}K^{-1}, $\sigma = 10^{-2}$ K^{-1} and $\kappa = 0.5$ W m^{-1}K^{-1}($\chi = 10^{-7}$m^{2}s^{-1}). In this case $t_p = 2\times10^{-5}$ s, $t_g = 6\times10^{-2}$ s, and our length unit is $l_d = 2$ mm. For a resonator with a large Fresnel number, the relaxation of the homogeneous the temperature is mainly due to the heat flux through the boundaries, and can be estimated from the Newton's colling law as $t_h \sim 10^1$ s. Then, under usual conditions we get the normalized decay times $\tau_p \sim 10^{-6}$, and $\tau_g \sim 10^{-2}$, and the diffusion constant $D \sim 10^0$.Note that for $\tau_{\rm g} = 1$ a single temperature field $(H + G)$ can be defined, whose evolution depends just on P and vice versa. This model exhibits a rich spatiotemporal dynamics and exhibits pattern formation and localized structure formed in the transverse cross-section of the cavity. Some of these results are studied in [5].

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